MATH 245 F19, Exam 2 Solutions

1. Carefully define the following terms: Proof by Contradiction, floor, Proof by Reindexed Induction.

   The Proof by Contradiction Theorem states: Let p, q be propositions. If $(p \wedge \neg q) \equiv F$, then $p \rightarrow q$ is true. Let $x \in \mathbb{R}$. Then there is a unique integer n, called the floor of x, satisfying $n \leq x < n+1$. To prove the proposition $\forall x \in \mathbb{N},\ P(x)$ by (reindexed) induction, we must (a) prove that $P(1)$ is true; and (b) prove that $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \rightarrow P(x)$.

2. Carefully define the following terms: Proof by Strong Induction, Fibonacci numbers, recurrence.

   To prove the proposition $\forall x \in \mathbb{N},\ P(x)$ by strong induction, we must (a) prove that $P(1)$ is true; and (b) prove that $\forall x \in \mathbb{N}$, $P(1) \wedge P(2) \wedge \cdots \wedge P(x) \rightarrow P(x+1)$. The Fibonacci numbers are a sequence given by $F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}$ $(k \geq 2)$. A recurrence is a sequence with the property that all but finitely many of its terms are defined in terms of its previous terms.

3. Let $a, b \in \mathbb{Z}$ with $b \geq 1$. Use minimum element induction to prove $\exists q, r \in \mathbb{Z}$ with $a = bq + r$ and $0 < r \leq b$.

   Let $S = \{m \in \mathbb{Z} : m \geq \frac{a}{b} - 1\}$, which is a nonempty set of integers. It has lower bound $\frac{a}{b} - 1$, so by minimum element induction it must have a minimum element, which we call q. Since $q \in S$, we have $q \in \mathbb{Z}$ and $q \geq \frac{a}{b} - 1$. Hence $bq \geq a - b$, which rearranges to $b \geq a - bq$. Set $r = a - bq$; by the above calculation $b \geq r$. Since q was minimal in S, $q - 1 \notin S$. Since $q \in \mathbb{Z}$ we must have $q - 1 < \frac{a}{b} - 1$, or $q < \frac{a}{b}$. We have $qb < a$, which rearranges to $0 < a - bq = r$. Combining, we have $0 < r \leq b$.

4. Let $x \in \mathbb{R}$. Prove that $\lfloor x \rfloor$ is unique; that is, prove that there is at most one $n \in \mathbb{Z}$ with $n \leq x < n+1$.

   Suppose there were two integers n, n', satisfying $n \leq x < n+1$ and also $n' \leq x < n'+1$. Combining $n \leq x$ with $x \leq n'+1$, we get $n < n'+1$. Combining $n' - 1 \leq x - 1$ with $x - 1 < n$, we get $n' - 1 < n$. Hence, we have $n' - 1 < n < n' + 1$. By a theorem from the book (1.12d), we must have $n = n'$.

5. Prove that, for every $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

   We prove by (ordinary) induction. The base case is $n = 1$: we have $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{n}{n+1}$. Now, let $n \in \mathbb{N}$ be arbitrary, and assume that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$. We add the next term, $\frac{1}{(n+1)(n+2)}$, to both sides, getting $\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$, as desired.

6. Solve the recurrence with $a_0 = 2, a_1 = 5$, and relation $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$. Hence we have a double root $r = 1$, and general solution $a_n = A1^n + Bn1^n = A + Bn$. Turning now to the initial conditions, we have $2 = a_0 = A + B \cdot 0 = A$, and $5 = a_1 = A + B \cdot 1 = A + B$. Hence $A = 2$ and $B = 3$, giving specific solution $a_n = 2 + 3n$.

7. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 5T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

In the notation of the Master Theorem, we have $a = 5, b = 2, c_n = n^2$. We have $c_n = \Theta(n^2)$, so $k = 2$. We set $d = \log_b a = \log_2 5$. Without a calculator, we can't find d exactly, but we know that $2 = \log_2 4 < d < \log_2 8 = 3$. Hence $2 < d < 3$, and in particular $d > k$. Hence we are in the "small c_n" case, and the Master Theorem tells us that $T_n = \Theta(n^d) = \Theta(n^{\log_2 5})$.

8. Prove or disprove: $\forall n \in \mathbb{Z},\ !m \in \mathbb{N}, n = m(4-m)$.

The statement is false. To disprove, we prove $\neg\forall n \in \mathbb{Z},\ !m \in \mathbb{N}, n = m(4-m)$, which is equivalent to $\exists n \in \mathbb{Z}, \exists m_1, m_2 \in \mathbb{N}, n = m_1(4-m_1) \wedge n = m_2(4-m_2) \wedge m_1 \neq m_2$. Take $n = 3, m_1 = 1, m_2 = 3$. We have $m_1 \neq m_2$, and $n = 3 = 1(4-1) = 3(4-3)$.

9. Prove that $n^2 - n = \Theta(n^2)$.

(easier part) We prove $n^2 - n = O(n^2)$. Take $n_0 = 1, M = 1$. For all $n \geq n_0$, $0 \leq n$ and hence $-n \leq 0$. Adding n^2 to both sides, we get $n^2 - n \leq n^2$, and thus $|n^2 - n| = n^2 - n \leq n^2 = M|n^2|$.

(harder part) We prove $n^2 - n = \Omega(n^2)$. Take $n_0 = 2, M = 2$. Let $n \geq n_0 = 2$. Multiplying by n, we get $n^2 \geq 2n$. Adding n^2, we get $2n^2 \geq n^2 + 2n$. Rearranging, we get $2n^2 - 2n \geq n^2$. Hence, $M|n^2 - n| = 2(n^2 - n) \geq n^2 = |n^2|$.

10. Prove that $\sqrt{5}$ is irrational.

We argue by contradiction. Suppose that $\sqrt{5}$ were rational. Then we would have $m, n \in \mathbb{Z}$, with $n \neq 0$, and $\sqrt{5} = \frac{m}{n}$. By cancelling any common factors, we may assume that m, n have no common factors. Squaring and rearranging gives $5n^2 = m^2$. Now, $5|m^2$, and 5 is prime, so $5|m$ (or $5|m$). Write $m = 5k$, for some integer k, and substitute back. We get $5n^2 = (5k)^2 = 25k^2$. Hence $n^2 = 5k^2$. Now $5|n^2$, and 5 is still prime, so $5|n$. Hence, m, n both have the common factor 5, a contradiction.